MEscope Application Note 01

The FFT, Leakage, and Windowing

The steps in this Application Note can be carried out using any MEscope package that includes the **VES-3600 Advanced Signal Processing** option. Without this option, you can still carry out the steps in this App Note using the **AppNote01** project file. These steps might also require MEscope software with a ***more recent release date***.

APP NOTE 01 PROJECT FILE

- To retrieve the Project for this App Note, **click here** to download **AppNote01.zip**

This Project file contains **numbered Hotkeys & Scripts** for carrying out the steps of this App Note.

- ***Hold down*** **the Ctrl key** and ***click*** **on a Hotkey** to display its Script window

INTRODUCTION

MEscope makes it very convenient to look at signals in either the time or the frequency domain. You can transform your data between these two domains at will without losing any of your data.

- **Transform | FFT** transforms a Data Block of **time domain waveforms** (**TWFs**) into their equivalent **Digital Fourier Transforms** (**DFTs**)
- **Transform | Inverse FFT** transforms a Data Block of **DFTs** into their equivalent **TWFs**

Another command in MEscope lets you create **DFTs**, **Auto & Cross Spectra**, **Power Spectral Densities (PSDs)**, and **Energy Spectral Densities (ESDs)**.

- **Transform | Spectra** creates a new Data Block of spectra from **TWFs** or other spectra

Additionally, MEscope has a command with which you can synthesize a broad variety of sine, random, chirp, or impact signals, plus Auto spectra.

- **File | New | Data Block** synthesizes a new Data Block of **TWFs** or **Auto spectra**

These commands are used in this App Note to review some basic properties of the **Fast Fourier Transform** (**FFT**) and learn how **common FFT leakage errors** can be minimized.

RULES FOR SAMPLED DATA

The **FFT** is an efficient algorithm for computing the **Discrete Fourier Transform (DFT)** of a uniformly-sampled **TWF**. Three basic formulas govern all calculations between uniformly-sampled **TWFs** and **DFTs**.

RULE #1 - TIME WAVEFORM SAMPLING

When a time domain signal is uniformly sampled. Its **total acquisition time (T)** equals the **time increment between successive samples (Δt)** multiplied by the **number of samples acquired (N).**

$$\boxed{T = (\Delta t)\, N}$$

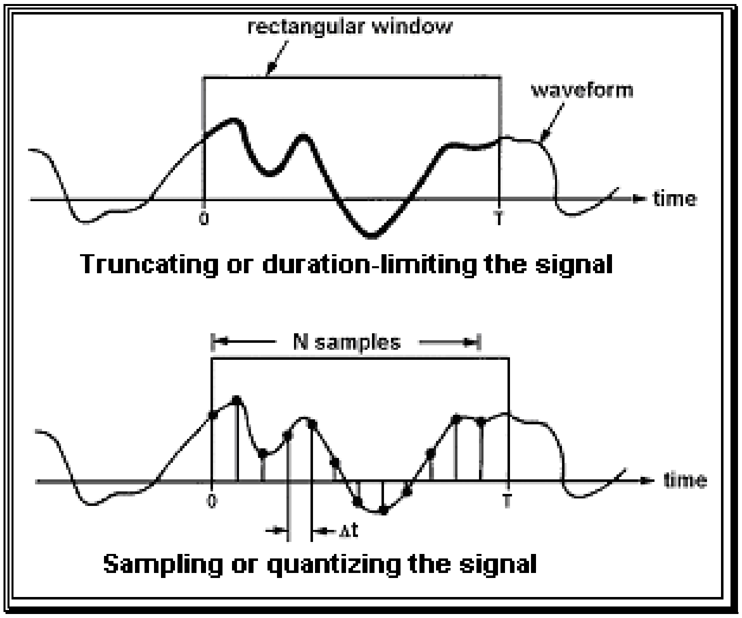
T ➔ time period of the sampled signal (in seconds)

Δt ➔ time step (or increment) between samples of data (in seconds)

N ➔ number of samples, also called the **Block Size**

The figure below depicts the sampling of a continuous (analog) time signal. This consists of two steps:

1. Multiply the signal by a **rectangular sampling window** (also called a **boxcar** window) to limit its duration to a **practical sampling time (T)**
2. Acquire the truncated time signal over a series of **equally spaced time increments**


rectangular window
waveform
0
T
time
Truncating or duration-limiting the signal
N samples
0
T
time
Δt
Sampling or quantizing the signal

Time Domain Windowing & Sampling.

The total time (**T**) of the sampled **TWF** includes the time **up to but not including** sample **N+1**.

RULE #2 - FREQUENCY SPECTRUM SAMPLING

The **DFT** of a uniformly-sampled time waveform is calculated with the Fast Fourier Transform (**FFT)** algorithm. The frequency span of a **DFT** is from **0 frequency (DC) to a maximum frequency** (**F_{max}**). **F_{max}** equals the **increment between adjacent samples (Δf)** multiplied by the **number of uniformly spaced complex-valued samples (N/2)**.

$$\boxed{F_{max} = \Delta f \ (N/2)}$$

F_{max} ➔ maximum frequency of the **DFT** (in Hz), also called the **Nyquist frequency**

Δf ➔ frequency increment between samples of the **DFT**

N/2 ➔ number of frequency samples, equal to **one half of the time waveform Block Size (N)**

- The **DFT** has **(N/2 + 1) samples** in it; a **DC** value plus **N/2** frequency samples
- The samples at **DC** and F_{max} are **real-valued**
- All other samples are **complex-valued**, having **real & imaginary** or **magnitude & phase** values
- A uniformly-sampled **TWF** is represented by (**N**) **real numbers**
- A uniformly-sampled **DFT** is represented by (**N/2**) **complex numbers**

RULE #3 - SHANNON'S SAMPLING THEOREM

The third formula that applies to uniformly-sampled data is called **Shannon's Sampling Theorem**.

Shannon's Sampling Theorem: The **maximum frequency (F_{max})** of a **DFT** equals **one-half the sampling rate** of its corresponding **TWF**.

$$\boxed{F_{max} = f_s/2}$$

$f_s = 1/\Delta t$ ➔ the **sample rate** (samples/second) at which the time waveform is sampled

Solving the frequency domain equation for **Δf**, and substituting for $2F_{max}$ gives a fourth equation:

$$\boxed{\Delta f = 2F_{max}/N = 1/N\Delta t = 1/T}$$

INCREASING THE FREQUENCY RESOLUTION

The equation above says that the frequency resolution (**Δf**) of a **DFT** is equal to the reciprocal of the **total acquisition time (T)** of the sampled time waveform. This relationship is important to remember when acquiring digital data with a digital acquisition system or a spectrum analyzer, or with a high-speed digital video camera.

Better Frequency Resolution: To obtain better frequency resolution (**smaller Δf**) in a **DFT**, its **TWF must be sampled over a longer time period (T)**, not at a higher sampling rate (f_s)

PERIODIC SQUARE WAVE

The Fourier series of a **Square wave of amplitude ± 1** and frequency (f_0) is,

$$y(t) = \frac{4}{\pi}\sum_{k=0}^{K}\frac{1}{(2k+1)}\mathrm{Sin}[2\pi(2k+1)f_0 t]$$

The first three terms of this series are:

$$y(t) = \mathrm{Sin}(2\pi f_0 t) + \frac{1}{3}\mathrm{Sin}(6\pi f_0 t) + \frac{1}{5}\mathrm{Sin}(10\pi f_0 t)$$

These three terms approximate a **square wave of amplitude (± π/4 = 0.785)** with a fundamental frequency (f_0).

A Data Block with three sine waves approximating a 5 Hz square wave was synthesized using the **File | New | Data Block** command. The figure above shows the values entered in the dialog box for this command.

Block Size (N) ➔ 100 Samples

Ending Value (F_{max}) ➔ 50 Hertz

The remaining four parameters are determined from the **Block Size & F_{max}.**

T ➔ 1 Second

Δt ➔ 0.01 Seconds

N/2 ➔ 50 Samples

Δf ➔ 1.00 Hertz

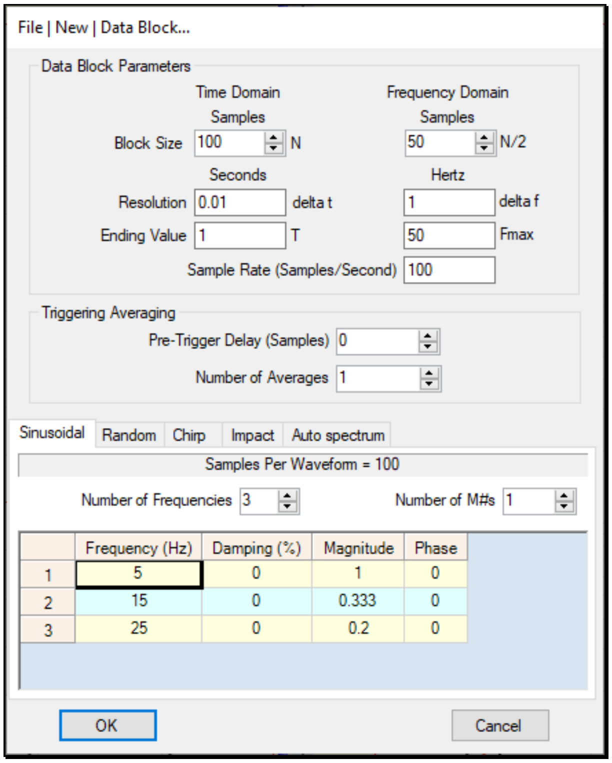
On the **Sinusoidal** tab,

Number of Frequencies ➔ 3

Number of M#s ➔ 1

Frequency values ➔ 5, 15 & 25 Hz

Magnitude values ➔ 1, $^1/_3$ & $^1/_5$



File | New | Data Block...

Data Block Parameters

	Time Domain Samples		Frequency Domain Samples	
Block Size	100	N	50	N/2
	Seconds		Hertz	
Resolution	0.01	delta t	1	delta f
Ending Value	1	T	50	Fmax

Sample Rate (Samples/Second) 100

Triggering Averaging

Pre-Trigger Delay (Samples) 0

Number of Averages 1

Sinusoidal | Random | Chirp | Impact | Auto spectrum

Samples Per Waveform = 100

Number of Frequencies 3 Number of M#s 1

	Frequency (Hz)	Damping (%)	Magnitude	Phase
1	5	0	1	0
2	15	0	0.333	0
3	25	0	0.2	0

OK Cancel

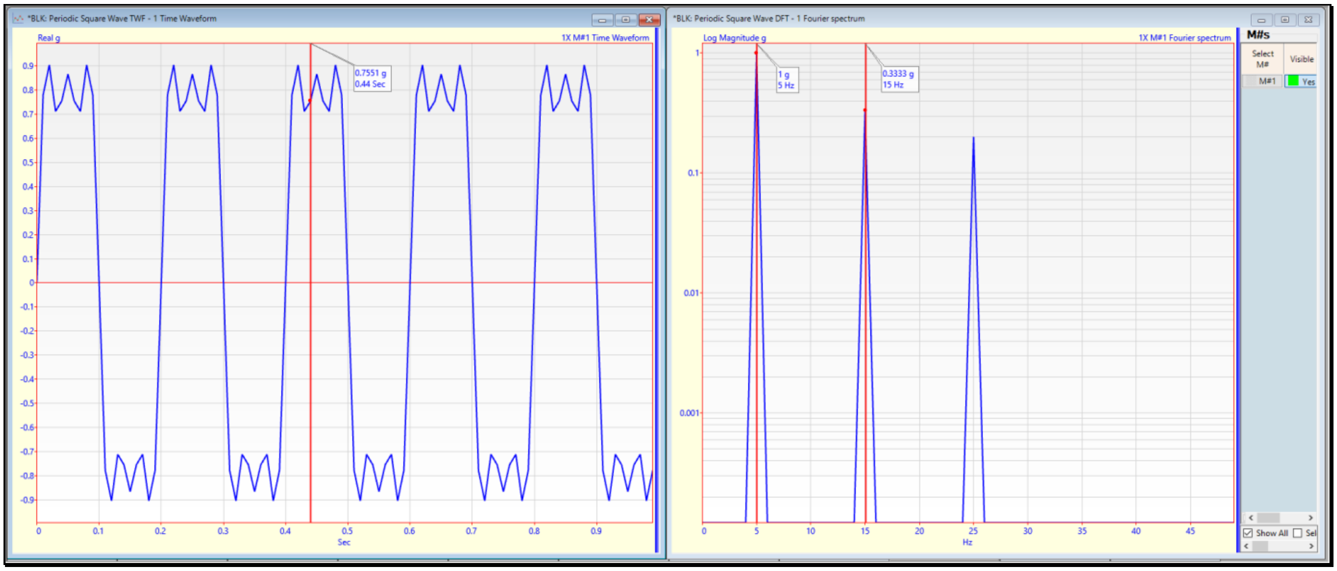
***File* | *New* | *Data Block* Entries to Synthesize a 5 Hz Square wave.**

STEP 1 - PERIODIC SQUARE WAVE

- ***Press* Hotkey 1 Periodic Square wave**

A Data Block **BLK: Periodic Square wave TWF** is displayed ***on the left***. This Data Block contains the the 5 Hz Square wave approximated by three sine waves. Its **DFT** is displayed in the Data Block **BLK: Periodic Square wave DFT *on the right***.

- The amplitude (in the center of the **square wave**) is **approximately ±0.795 g**, as expected
- **Exactly five cycles** of this periodic waveform are captured in the **1 second** duration of its sampling window

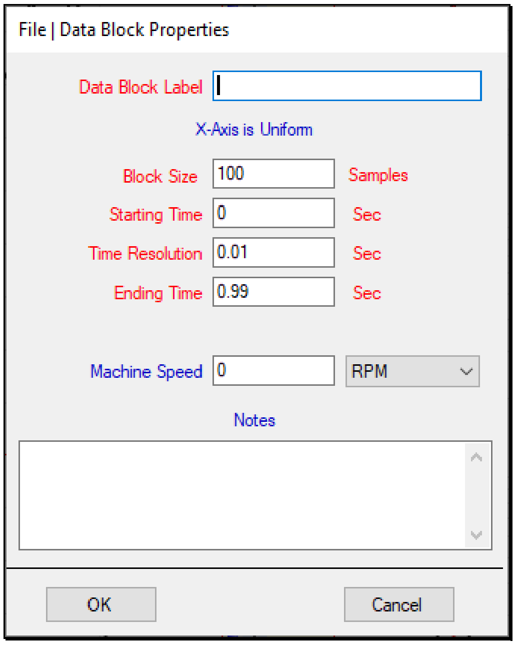


*5 Hz Square Wave **TWF** on the Left and Its **DFT** on the Right.*

- The **Band** cursor in the **DFT** shows the magnitudes of the **(1.0 g) 5 Hz & (0.33 g) 15 Hz** sine waves
- ***Drag*** the Band cursor to the **15 & 25 Hz** peaks in the **DFT**
- Now the **Band** cursor in the **DFT** shows the magnitudes of the **(0.33 g) 15 Hz & (0.20 g) 25 Hz** sine waves

To verify that the properties of the **TWF** agree with the **N**, **T**, **ΔT,** & the **Starting Time** of the synthesized square,

- ***Right click*** in the graphics area of **BLK: Periodic Square wave TWF** and execute **File | Data Block Properties**



File | Data Block Properties

Data Block Label

X-Axis is Uniform

Block Size 100 Samples

Starting Time 0 Sec

Time Resolution 0.01 Sec

Ending Time 0.99 Sec

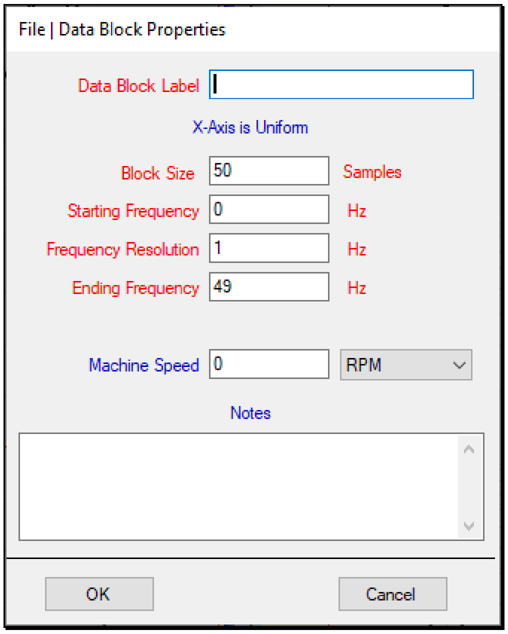
Machine Speed 0 RPM

Notes

***File | Data Block Properties** of **BLK: Periodic Square Wave TWF**.*

To verify that the properties of the **DFT** agree with the **N/2**, **F$_{\mathbf{max}}$**, & **Δf** of the synthesized square wave,

- ***Right click*** in the graphics area of **BLK: Periodic Square wave DFT** and execute **File | Data Block Properties**



File | Data Block Properties

Data Block Label

X-Axis is Uniform

Block Size 50 Samples

Starting Frequency 0 Hz

Frequency Resolution 1 Hz

Ending Frequency 49 Hz

Machine Speed 0 RPM

Notes

***File | Data Block Properties** Box of **BLK: Periodic Square wave DFT**.*

LEAKAGE - A FUNDAMENTAL FFT PROBLEM

In the preceding step, a synthesized square wave was used that contained three sinusoidal components at **5**, **15** and **25** Hz. The square wave was synthesized over a time period (**T)** of **1** second.

- Each of the square wave components **completed exactly an integer number of cycles (5, 15 & 25 respectively)** within the **1-second** sampling window
- Each resulting peak in the **DFT was on an integer multiple of Δf** (**1.00 Hz**)

When **T is exactly** equal to **N cycles of a periodic signal**, the signal is said to be **periodic in its sampling window**.

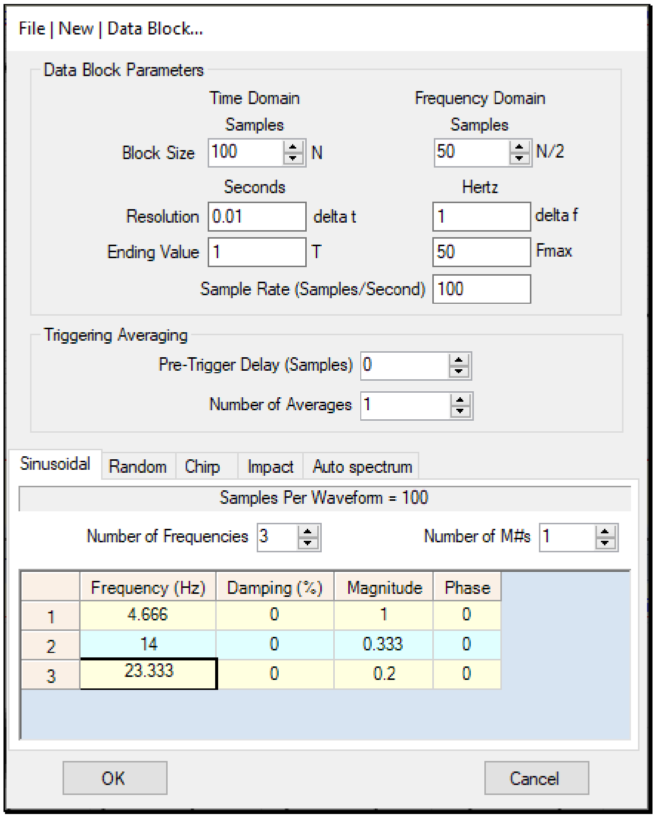
Frequencies & magnitudes of the periodic components in a **TWF** are precisely represented in its **DFT**.

STEP 2 - NON-PERIODIC SQUARE WAVE

- ***Press*** **Hotkey 2 Non-Periodic Square wave**

For this step, the frequency of the approximate square wave **was changed slightly**. A Data Block with three sine waves approximating a **$4^{2}/_{3}$** Hz square wave was synthesized using the **File | New | Data Block** command. The figure below shows the values entered in the dialog box for this command.

- The frequencies of the three sine waves were changed from **5, 15 & 25 Hz** to **4.666**, **14** & **23.333 Hz**
- **4.666**, **14** & **23.333 Hz** are **1, 3 & 5** times the fundamental **$4^{2}/_{3}$ Hz** frequency respectively



File | New | Data Block...

Data Block Parameters

Time Domain Samples: Block Size 100 N

Frequency Domain Samples: 50 N/2

Seconds: Resolution 0.01 delta t; Ending Value 1 T

Hertz: 1 delta f; 50 Fmax

Sample Rate (Samples/Second) 100

Triggering Averaging

Pre-Trigger Delay (Samples) 0

Number of Averages 1

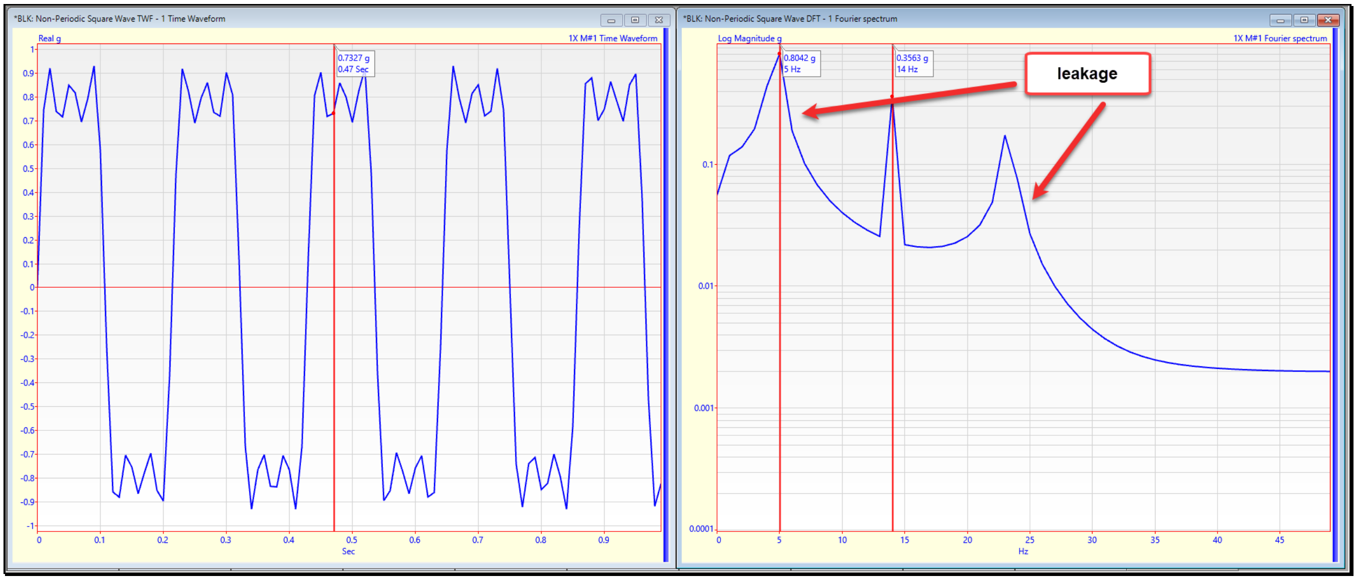
Samples Per Waveform = 100

Number of Frequencies 3

Number of M#s 1

	Frequency (Hz)	Damping (%)	Magnitude	Phase
1	4.666	0	1	0
2	14	0	0.333	0
3	23.333	0	0.2	0

File* | *New* | *Data Block *Entries for a 4 $^{2}/_{3}$ Hz Square wave.*



*4 $^2/_3$ Hz Square wave **TWF** on the Left and Its **DFT** on the Right.*

A Data Block **BLK: Non-Periodic Square wave TWF** is displayed ***on the left***. This Data Block contains the the **4$^2/_3$ Hz** square wave that was synthesized using three sine waves. Its **DFT** is displayed in the Data Block **BLK: Non-Periodic Square wave DFT *on the right***.

- The **4 $^2/_3$ Hz** square wave **TWF looks like** the **5 Hz** square wave **TWF**
- The amplitude (in the center of the **TWF**) is **approximately ±0.795 g**, as expected
- The **third harmonic (14 Hz)** is **an exact integer multiple of Δf** and **completes exactly 14 cycles** within the 1 second sampling window
- The **first & fifth harmonics** are **non-periodic in the sampling window** and their frequency peaks are widened in the **DFT**

The **4 $^2/_3$** Hz **TWF** exhibits the **same shape and amplitude range** as the **5 Hz TWF**, but only **4 $^2/_3$** cycles are contained within its sampling window instead of **5** cycles.

There is a **significant difference** between the **DFT** of the **4 $^2/_3$** Hz square wave and the **DFT** of the **5** Hz square wave.

The peaks at **4 $^2/_3$** Hz & **23.333 Hz** have leakage surrounding them!

While the three sinusoidal peaks are still evident in the **DFT** of the **4 $^2/_3$ Hz** square wave, the rest of the spectrum is quite different from the **5 H**z square wave. There is significant energy **at all frequencies**, even though **only three well-separated *tones*** were used to create the non-periodic signal.

- The ***magnitudes*** of the 1X and 5X peaks in the **DFT are less than** the correct values of **1.0** & **0.2**
- Only the **14** Hz **periodic** 3X peak has a magnitude **close to the correct value** (**0.333**)

WHAT CAUSES LEAKAGE?

Whenever the **FFT** is applied to a signal that is **not exactly periodic in its sampling window**, leakage will occur in its **DFT**.

- Leakage occurs when two signals are multiplied together in one domain, which is **equivalent to convolving** their **DFTs** together in the other domain
- Convolution is a process of **shifting and adding together** two signals, so that the resultant **DFT** is a **smeared version** of the expected **DFT**
- Vibrant technical paper **[1]** has more details on convolution and leakage

In Steps 1 & 2, a square wave was multiplied by a **Rectangular** sampling window. The resultant **DFT** was the convolution of the **DFT** of the square wave with the **DFT** of the **Rectangular** window.

- In Step 1, the **DFT** of the periodic square wave convolved with the **DFT** of the **Rectangular** window **caused no smearing** of the **DFT** of the periodic square wave
- In Step 2, the **DFT** of the non-periodic square wave convolved with the **DFT** of the **Rectangular** window **caused smearing** of the **DFT** of the non-periodic signal, or leakage of the **DFT** around the peaks of the non-periodic sine waves

ANOTHER EXPLANATION OF LEAKAGE

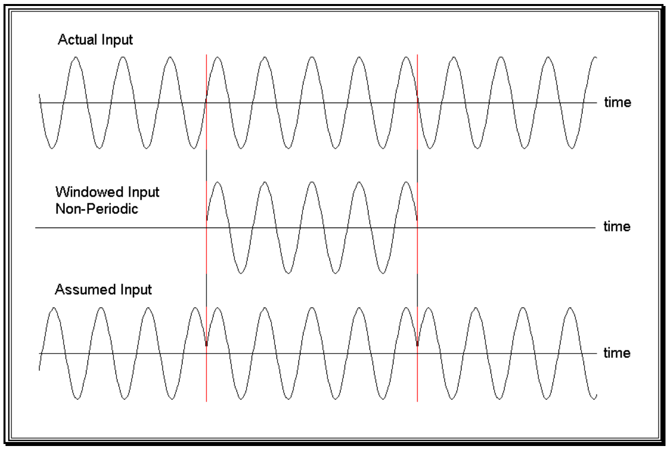
Another way to understand leakage is to look at what happens to a sampled signal.

Fundamental Rule of the FFT: Sampling of a signal in one domain causes repetition of its Fourier transform in the other domain.

The **FFT** computes the **DFT** (forward **FFT**) or the **TWF** (inverse **FFT**), **assuming that is it repeated outside of its sampling window**.

When a **signal is non-periodic** in its sampling window, instead of getting the **FFT** of a continuous sine wave, **a smeared version** of its spectrum results.

This phenomenon is illustrated below. The actual signal is a continuous sine wave. The finite duration of the windowed version **is truncated at both ends** by the finite length of the sampling window.



Repetition of a Non-Periodic Signal Outside its Sampling Window.

Another rule for obtaining a leakage-free signal is.

Leakage-Free Signal: Any signal that is **completely contained within its sampling window**, (like a transient that damps out within the sampling window), is also periodic in its sampling window, and **has a leakage-free DFT**.

REDUCING LEAKAGE

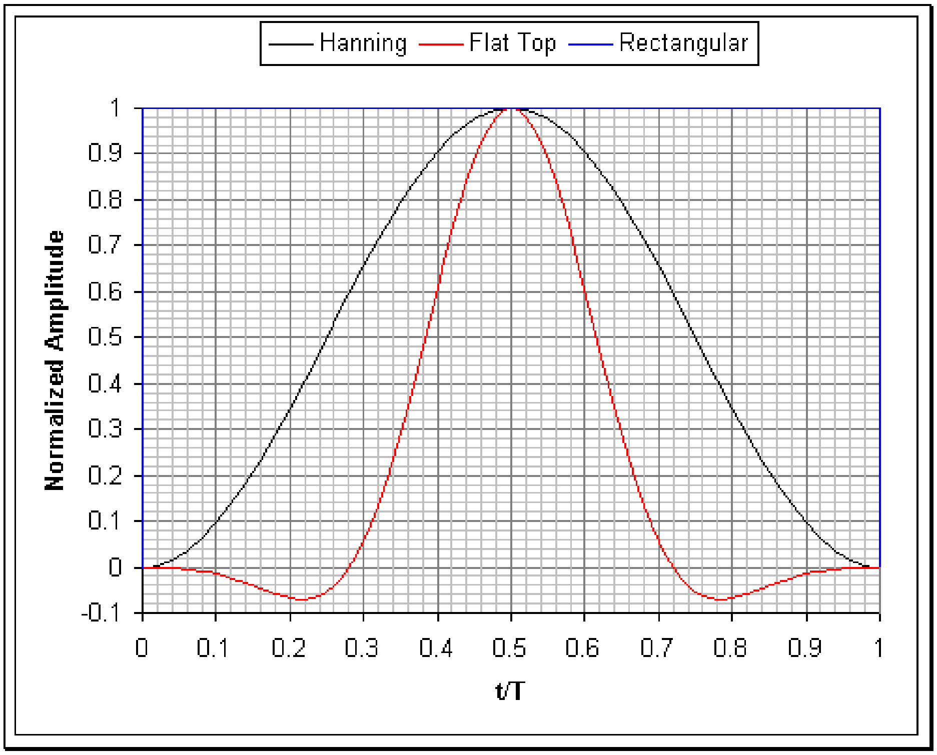
The Bad News: If a signal is **non-periodic or not completely contained** within its sampling window, **leakage** in its spectrum **cannot be eliminated**.

However, leakage **can be significantly reduced** so that the resulting **DFT** contains a more accurate representation of the spectrum of the signal.

The Good News: Leakage can be reduced by multiplying a sampled signal by a **special windowing function** before it is transformed using the **FFT**.

MEscope has three built-in time domain windowing functions, a **Rectangular**, a **Hanning**, and a **Flat Top** window.

- Unless another windowing function is specified, a **Rectangular** window is **applied by default** to any sampled signal within its sampling window



Window shapes: Rectangular, Hanning and Flat Top.

RECTANGULAR WINDOW

The **Rectangular** window is the proper choice for analyzing any signal that **is periodic within**, or that **is completely contained within** the time duration **(T)** of its sampling window.

The Rectangular window is also the proper choice when analyzing periodic signals using **order-normalized** sampling, a process by which the sampling rate (**f$_s$**) **is changed** in proportion to the operating speed of a rotating machine.

If a Rectangular window is applied to any signal that is non-periodic, not completely contained within, or not order-normalized within its sampling window, the DFT of that signal will contain leakage

HANNING WINDOW

The **Hanning** window is the best window choice when analyzing **broad-band random** signals, **non-periodic cyclic** signals, or a mixture of the two.

The **Hanning** window has excellent ability to separate closely spaced frequencies and tones buried in a noisy background.

FLAT TOP WINDOW

The **Flat Top** window gives **improved amplitude precision** of the peaks in the **DFT** of **cyclic or narrow-band** signals. It provides an accurate order-based ODS from machine order peaks. However, the **Flat Top** window widens the peaks in a spectrum, so it is not good for decerning closely spaced peaks.

- Both the **Hanning** & **Flat Top** windows can be used to improve the spectrum analysis of ***non-periodic*** and ***non-transient*** signals
- Both the **Hanning** & **Flat Top** windows greatly distort the shape of a **TWF**, but both are **highly beneficial for reducing leakage** on the resulting **DFT**

FFT DIGITAL FILTERS

Leakage in a **DFT** can be viewed from yet another perspective.

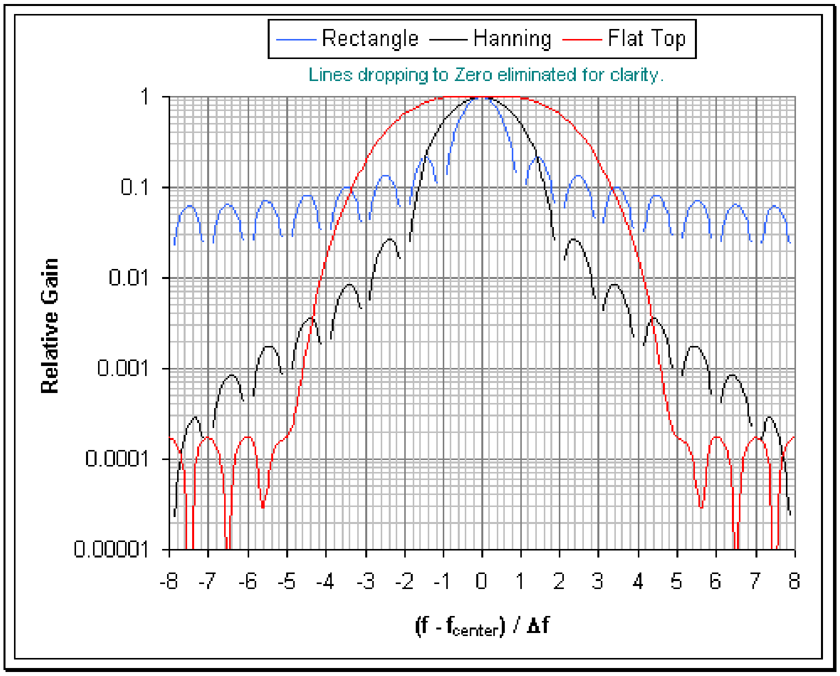
The **FFT** numerically implements a bank of parallel digital filters.

Each filter has the **same constant bandwidth** (nominally **Δf**).

The filter center-frequencies are spaced **Δf** apart.

In an ideal world, the transfer function of each **FFT** digital filter would be a **"brick wall"** rectangle of width **Δf**. However, such ideal filters do not exist, either **physically** or **mathematically**. Instead, the transfer function of each **FFT** filter has a unique shape. The filter shapes for the **Rectangular**, **Hanning** and **Flat Top** windows are shown below.

The **FFT** filter for each of these windows is far broader than one **Δf**. Each one exhibits a maximum gain (sensitivity) of **"1"** at the **center frequency** (**f_{center}**). Adjacent filter center frequencies are spaced **Δf** apart, but **considerable overlap exists** in a bank of these filters.



*Magnitude of **FFT** Filter Transfer Functions.*

The **Rectangular** filter has the narrowest center lobe, (a desirable attribute) spanning **2Δf's** between adjacent zeros. Outside of the center lobe, the **Rectangular** window exhibits **zero gain at every multiple of Δf**.

Compared to the **Hanning & Flat Top** window filters, the **Rectangular** window has the highest side lobe amplitudes and the greatest curvature across the center lobe.

The **Rectangular** window works best when applied to **signals that are periodic** in their sampling window. Each **periodic** signal component exists at **exactly the frequency of one of the zeros** of the Rectangular window filter.

- When a **single frequency tone** moves slightly off the center frequency of a Rectangular filter by (±**Δf / 2** or less), it **no longer coincides with the zeros of the adjacent filters** of the Rectangular window filter

The **Hanning** window has side lobes with heights that are **much less in amplitude** than the side lobes of the **Rectangular** window. This reduced side lobe gain **minimizes the leakage** of a signal into adjacent frequencies.

However, the Hanning window has a broader center lobe **with 4 Δf's between its nearest bounding zeros**.

The frequency resolution of a tone is reduced in its **DFT** when a Hanning window is used.

As its name suggests, the **Flat Top** window has a **very wide center lobe**.

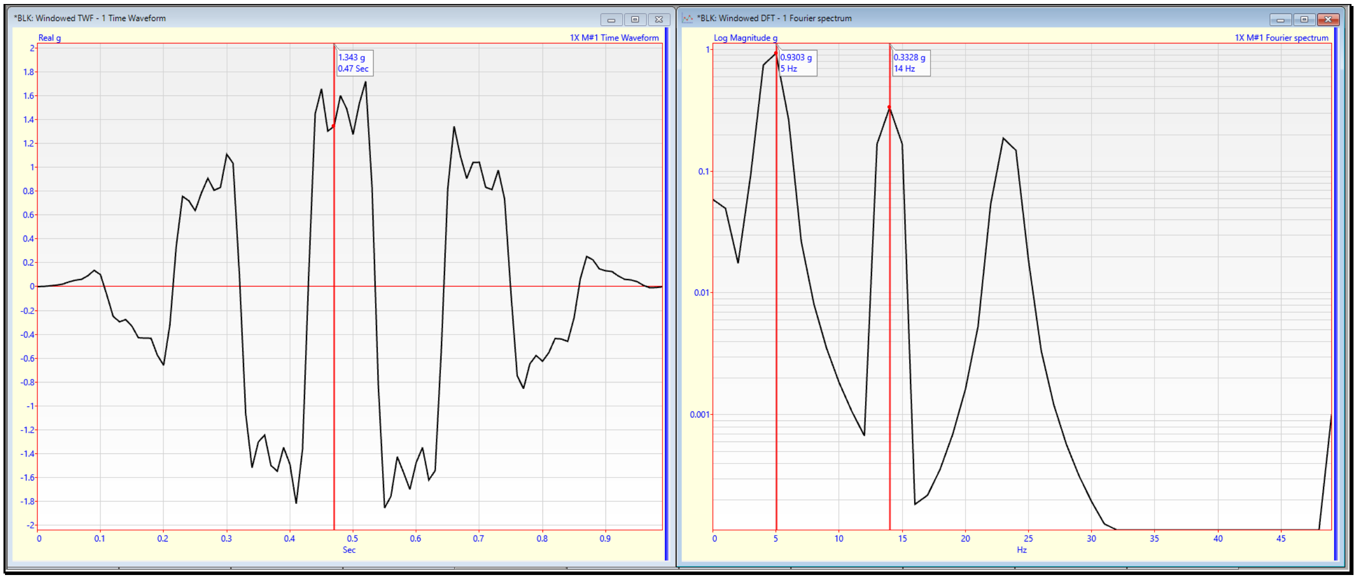
The center lobe of the **Flat Top** is **extremely flat** (within ± 0.1 %), providing an accurate amplitude even when a tone varies as much as ±**Δf / 2** from the center $\mathbf{f_{center}}$ of its digital filter. The side lobes of the **Flat Top** filter are also well suppressed, but they are of nearly equal amplitude.

STEP 3 - HANNING WINDOW APPLIED TO THE NON-PERIODIC SQUARE WAVE

- ***Press* Hotkey 3 Hanning Window Applied to Non-Periodic Waveform**

When **Hotkey 3 is *pressed***, a **Hanning** window is applied to the **4 $^2/_3$ Hz non-periodic** square wave before it is transformed into its **DFT**.

The Data Block **BLK: Windowed Non-Periodic Waveform** is displayed ***on the left***. This Data Block contains the the **4$^2/_3$ Hz** square wave with a Hanning window applied to it. Its **DFT** is displayed in the Data Block **BLK: Windowed Non-Periodic Waveform DFT *on the right***.



*Hanning Windowed 4 $^2/_3$ Hz square wave **TWF** on the Left and Its **DFT** on the Right.*

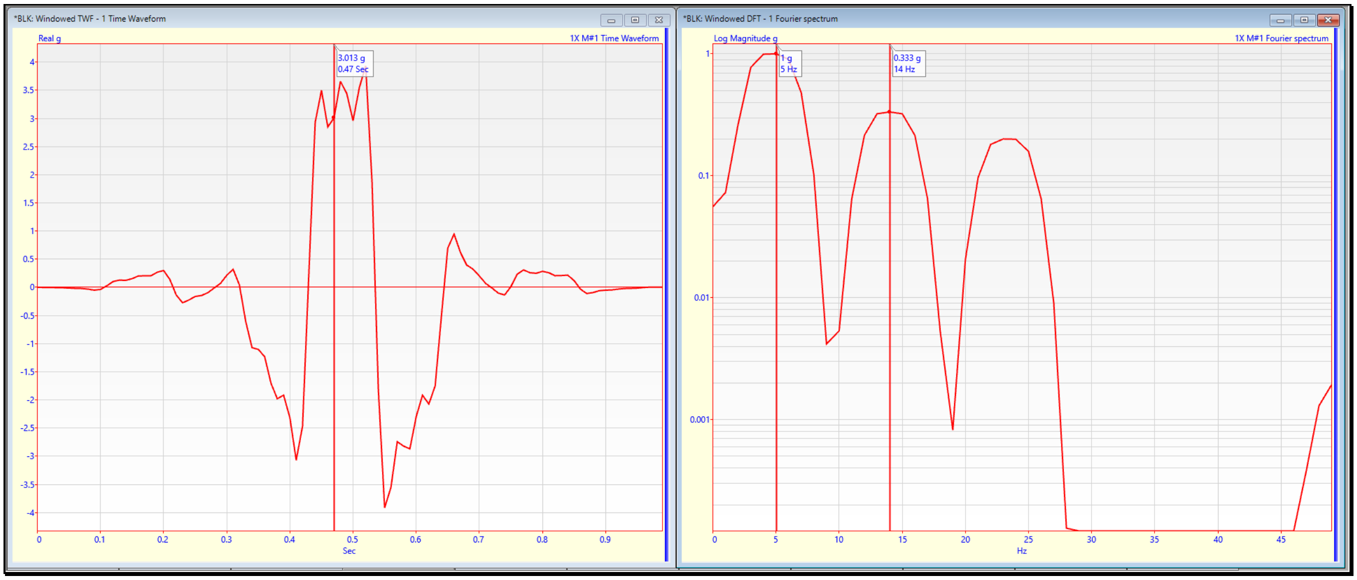
The windowed **TWF** is completely distorted, by the three sinusoidal **peaks** of the **DFT** are **clearly distinguishable**, and their amplitudes are **close to the correct values**.

STEP 4 - FLAT TOP WINDOW APPLIED TO THE NON-PERIODIC SQUARE WAVE

- ***Press* Hotkey 4 Flat Top Window Applied to Non-Periodic Waveform**

When **Hotkey 4 is *pressed***, the **Flat Top** window is applied to the **4 $^2/_3$ Hz non-periodic** square wave before it is transformed into its **DFT**.

A Data Block **BLK: Windowed Non-Periodic Waveform** is displayed ***on the left***. This Data Block contains the the **4$^2/_3$ Hz** square wave with a Flat Top window applied to it. Its **DFT** is displayed in the Data Block **BLK: Windowed Non-Periodic Waveform DFT *on the right***.



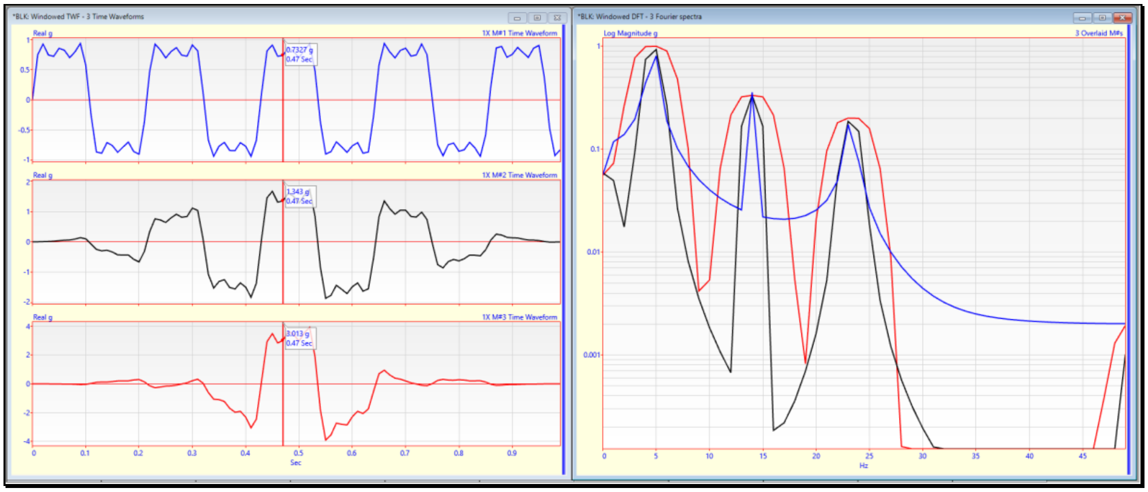
*Flat Top Windowed 4 $^2/_3$ Hz Square wave **TWF** on the Left and Its **DFT** on the Right.*

The windowed **TWF** is completely distorted, by the three sinusoidal **peaks** of the **DFT** are **correct**, and their amplitudes are **precisely correct**.

STEP 5 - WINDOW COMPARISON

- ***Press* Hotkey 5 Comparison of Windowed Waveforms**

When **Hotkey 5 is *pressed***, the **Rectangular, Hanning** and **Flat Top** windows are applied to the **4 $^2/_3$ Hz non-periodic** square wave. The three windowed **TWFs** are displayed ***on the left***, and their **DFTs** are overlaid ***on the right***.



4 $^2/_3$ Hz Square Wave With Rectangular, Hanning, Flat Top Windows Applied.

This comparison clearly shows that either the **Hanning** or **Flat Top** window **minimized the leakage** in the **DFT**, but did not eliminate it.

- The **Flat Top** window provided the **best magnitude estimates** compared to the other two windows, as shown in the following table.
- The **periodic 14 Hz** component **did not add leakage** to the **4.6667 Hz** or **23.333 Hz** peaks, but the magnitude at **14 Hz** was affected by the leakage from the other two frequency components.

Peak	Rectangular	Hanning	Flat Top
4.666 Hz	-19.50%	-6.95%	0.00%
14.0 Hz	7.01%	-0.07%	0.00%
23.333 Hz	-13.85%	-6.95%	0.00%

Percent Amplitude Error Using Different Windows.

STEP 6 - REVIEW STEPS

To review the steps of this App Note,

- ***Press* Hotkey 6 Review Steps**

REFERENCES

Richardson, M., ***Fundamentals of the Discrete Fourier Transform***, Sound and Vibration Magazine, March 1978.

Richardson, M., ***Modal Analysis Using Digital Test Systems***, Seminar on Understanding Digital Control and Analysis in Vibration Test Systems, Shock and Vibration Information Center Publication, Naval Research Laboratory, Washington, D.C., May 1975.